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On the integrability and singularity structure aspects of deformed nonlinear evolution equations of AKNS type

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Abstract. In this paper, a singularity structure analysis of some important inhomogeneous nonlinear evolution equations (NLEEs) of AKNS type introduced by Burtsev, Zakharov and Mikhailov (BZM) is carried out and they are shown to possess the Painlevé property. The other integrability properties such as Lax pair, Bäcklund transformation and soliton solutions of these systems are also brought out in detail from the Painlevé analysis. We also point out that the non-Painlevé nature of the system of partial differential equations satisfied by the variable spectral parameter may not affect the Painlevé property and hence the integrability of the associated NLEEs.

1. Introduction

The study of wave propagation in an inhomogeneous medium and the associated nonlinear partial differential equations (PDEs) have assumed greater significance ever since the identification of solitons in them [1, 2]. In the above nonlinear PDEs, the spectral parameter is regarded as a variable quantity that satisfies an overdetermined system of partial differential equations (PDEs) which is uniquely determined by the auxiliary linear problem. By considering the variable spectral parameter, Burtsev *et al* [3] have generated the deformations of various well known integrable equations and have also proposed that to every equation with a constant spectral parameter to which the scheme of inverse scattering method [4] is applicable, there corresponds an entire class of equations with a variable spectral parameter. They have also noted that the system of PDEs satisfied by the variable spectral parameter may not have the Painlevé property in general and that these systems may be integrable in quite a new sense. In this paper, we take up those deformed systems of AKNS type discussed explicitly by Burtsev *et al* [3] in the appendix of their paper and investigate their singularity structure. We prove that all these AKNS type equations satisfy the Painlevé property [5]. We also deduce other integrability properties like the Lax pair, the Bäcklund transformation (BT) and soliton solutions from the Painlevé analysis. We also suggest that the non-Painlevé nature of the system of PDEs of the variable spectral parameter may not affect the Painlevé property of the deformed PDEs and hence their integrability.

The plan of the paper is as follows. In section 2, we investigate the singularity structure aspects of the deformed MKdV equation and prove its Painlevé property. We also obtain its Lax pair and soliton solutions. Section 3 is concerned with the deformation of the nonlinear Schrödinger (NLS) equation. In particular, we concentrate on the singularity structure aspects of cylindrical NLS equation. Other deformations of the NLS equation such as the linearly

x-dependent and radially symmetric NLS equations are also briefly mentioned. Section 4 is devoted to a study of the deformed KdV system. A very brief mention of the deformed Kaup system is also made. Finally, we conclude with a brief discussion of results in section 5 and point out that the Painlevé property of the deformed PDEs is not affected by the Painlevé property of the equation satisfied by the variable spectral parameter with the Maxwell–Bloch equation as an example.

2. Deformed modified Korteweg–deVries (MKdV) equation

The deformed MKdV equation given in the paper by Burtsev *et al* [3] (their equation (A.5)) has the form

$$u_t + (xu)_{xxx} = \pm 2\left[\left(x\int u^2\,\mathrm{d}x\right)_x u\right]_x. \tag{1}$$

In order to analyse the singularity structure [5] of this non local equation, we introduce the transformation

$$u^2 = v_x \tag{2}$$

so that equation (1) can be rewritten as a set of coupled nonlinear PDEs of the form

$$u_t + 3u_{xx} + xu_{xxx} = \pm[4uv_x + 2vu_x + 6xu_xv_x] \tag{3a}$$

$$u^2 = v_x. \tag{3b}$$

2.1. The singularity structure analysis

We consider first the positive sign on the right-hand side of equation (3a) and assume the leading orders of the local Laurent expansion in the neighbourhood of a movable non-characteristic singular manifold $\phi(x,t)$ to have the form

$$u = u_0\phi^{\alpha} \qquad v = v_0\phi^{\beta} \qquad (\phi_x, \phi_t \neq 0) \tag{4}$$

where u_0, v_0 are analytic functions of (x,t) and α and β are integers to be determined. Substituting (4) into (3) and balancing the nonlinear terms against the most dominant linear terms, we get

$$\alpha = \beta = -1 \tag{5}$$

with

$$v_0 = -\phi_x \qquad u_0^2 = \phi_x^2. \tag{6}$$

For finding resonances, that is powers at which arbitrary functions can enter into the Laurent series, we now substitute

$$u = u_0\phi^{-1} + \cdots + u_j\phi^{j-1} + \cdots \qquad v = v_0\phi^{-1} + \cdots + v_j\phi^{j-1} + \cdots \tag{7}$$

into (3) and equate the coefficients of ϕ^{j-4} and ϕ^{j-2} to get

$$\begin{bmatrix} (j-1)[(j-2)(j-3)\phi_x^3 + 6v_0\phi_x^2] & 6(j-1)u_0\phi_x^2 \\ 2u_0 & -(j-1)\phi_x \end{bmatrix}\begin{bmatrix} u_j \\ v_j \end{bmatrix} = 0. \tag{8}$$

Evaluating equation (8) with (6), we obtain the resonances as

$$j = -1, 1, 3, 4. \tag{9}$$

The resonance at $j = -1$ naturally represents the arbitrariness of the singular manifold $\phi(x, t) = 0$. In order to prove the existence of arbitrary functions at the other resonance values, we now substitute the full Laurent expansion

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-1} \qquad v = \sum_{j=0}^{\infty} v_j \phi^{j-1}$$

into the equation (3). Collecting the coefficients of (ϕ^{-3}, ϕ^{-1}), we obtain

$$6u_0\phi_x^2 + 6xu_{0x}\phi_x^2 + 6xu_0\phi_x\phi_{xx} = 4u_0\phi_x^2 + 6xu_{0x}\phi_x^2 + 6xu_0\phi_x\phi_{xx} + 2u_0\phi_x^2 \tag{10a}$$

$$2u_0u_1 = v_{0x}. \tag{10b}$$

From the above set of equations (10*a*,*b*), it is evident that v_1 is arbitrary. Similarly from the coefficients of (ϕ^{-2}, ϕ^0), we obtain

$$u_2 = \frac{1}{6x\phi_x^3}[u_0\phi_t + 4u_{0x}\phi_x - u_0\phi_{xx} + 3xu_{0xx}\phi_x - 3xu_{0x}\phi_{xx} + xu_0\phi_{xxx} + 4\phi_x^2u_1 + 6xu_{1x}\phi_x^2 - 2u_0\phi_xv_1 - 6xu_0\phi_xu_1^2] \tag{11a}$$

$$v_2 = \frac{-1}{6x\phi_x^3}[6x\phi_x^2u_1^2 + 6x\phi_x^2v_{1x} - 2\phi_x\phi_t - 8u_0u_{0x} + 2\phi_x\phi_{xx} - 6xu_0u_{0xx} + 6x\phi_{xx}^2 - 2x\phi_x\phi_{xxx} - 8u_0u_1\phi_x - 12x\phi_xu_0u_{1x} + 4\phi_x^2v_1]. \tag{11b}$$

Again, by collecting the coefficients of (ϕ^{-1}, ϕ^1), we have

$$\begin{bmatrix} 2u_0 & -2\phi_x \\ -12x\phi_x^3 & 12xu_0\phi_x^2 \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \tag{12a}$$

where

$$A = v_{2x} - 2u_1u_2 \tag{12b}$$

$$B = -u_{0t} - 3u_{0xx} - xu_{0xxx} + 4u_0(v_{1x} + v_2\phi_x) + 4u_1v_{0x} + 4u_2\phi_x^2 + 6x[u_{0x}(v_{1x} + v_2\phi_x) - u_0\phi_xv_{2x} - \phi_{xx}(u_{1x} + u_2\phi_x) + u_{2x}\phi_x^2] + 2[u_{0x}v_1 - u_0v_2\phi_x - \phi_x(u_{1x} + u_2\phi_x)]. \tag{12c}$$

From (12), one can easily check that the two linear equations degenerate into a single equation and hence one of the coefficients u_3 or v_3 is arbitrary. Similarly from the coefficients of (ϕ^0, ϕ^2), one can show that either u_4 or v_4 is arbitrary. Thus, the general solution $\{u, v\}(x, t)$ of equation (3) admits the required number of arbitrary functions without the introduction of any movable critical manifold, thereby satisfying the Painlevé property and hence the system is expected to be integrable.

Similarly, by considering the negative sign on the right-hand side of equation (3*a*) and proceeding in a similar fashion, the Painlevé property can be easily verified.

2.2. *Lax pair*

As equation (3) satisfies the Painlevé property, we shall now proceed to obtain the integrability properties associated with it. To construct the Lax pair, the Laurent expansion (7) is truncated at the constant level term as

$$u = u_0\phi^{-1} + u_1 \qquad v = v_0\phi^{-1} + v_1 \tag{13}$$

where u_0 and v_0 satisfy equation (6). The above condition in fact can be treated as an auto Bäcklund transformation of equation (3) if both the sets (u, v) and (u_1, v_1) satisfy the evolution equation (3). Substituting (13) into (3) and equating different powers of ϕ, we obtain the following set of differential equations.

(ϕ^{-4}, ϕ^{-2}):

$$u_0^2 = \epsilon\phi_x^2 \qquad v_0 = -\epsilon\phi_x. \tag{14}$$

(ϕ^{-3}, ϕ^{-1}):

$$6u_0\phi_x^2 + x[6u_{0x}\phi_x^2 + 6u_0\phi_x\phi_{xx}] = \epsilon[-6u_0v_0\phi_x - 6x\phi_x[u_0v_0]_x] \tag{15a}$$

$$2u_0u_1 = v_{0x}. \tag{15b}$$

(ϕ^{-2}, ϕ^0):

$$\begin{aligned}&-u_0\phi_t - 6u_{0x}\phi_x - 3u_0\phi_{xx} - 3xu_{0xx}\phi_x - 3xu_{0x}\phi_{xx} - xu_0\phi_{xxx}\\ &= \epsilon[4u_0v_{0x} - 4v_0\phi_xu_1 + 6xu_{0x}v_{0x} - 6xu_0\phi_xv_{1x} - 6xu_{1x}v_0\phi_x\\ &\quad +2u_{0x}v_0 - 2u_0\phi_xv_1]\end{aligned} \tag{16a}$$

$$u_1^2 = v_{1x}. \tag{16b}$$

(ϕ^{-1}):

$$\begin{aligned}&u_{0t} + 3u_{0xx} + xu_{0xxx} = \epsilon[4(u_0v_{1x} + u_1v_{0x}) + 6x(u_{0x}v_{1x} + u_{1x}v_{0x})\\ &\quad +2u_{0x}v_1 + 2u_{1x}v_0]\end{aligned} \tag{17}$$

where $\epsilon = \pm 1$.

Now considering (14*a*) and differentiating with respect to time t, we have

$$u_0u_{0t} = \epsilon\phi_x\phi_{xt}. \tag{18}$$

Evaluating u_{0t} from equation (17) and ϕ_{xt} from equation (16*a*) (equation (15*a*) is just an identity and (15*b*) and (16*b*) define u_1 and v_1 in terms of ϕ) and substituting them in equation (18), we are now led to the following equation after simplification,

$$u_{0x}^2 - 2u_1u_0\phi_{xx} - 2u_0u_{1x}\phi_x - u_0u_{0xx} + 2u_1u_{0x}\phi_x = 0. \tag{19}$$

This equation can be simplified as

$$\left[\frac{u_{0x} + 2u_1\phi_x}{u_0}\right]_x = 0 \tag{20}$$

where we have made use of the fact that $u_0\phi_{xx} = u_{0x}\phi_x$ as can be seen from equation (14). In order to derive the linear eigenvalue problem from equation (20), we make use of the following transformation

$$u_0 = \sqrt{\epsilon a_0b_0} \qquad u_1 = \epsilon\frac{(a_0b_1 + b_0a_1)}{\sqrt{\epsilon a_0b_0}}. \tag{21}$$

Substituting equation (21) in (20), we obtain after simplification

$$\left[\frac{a_{0x} + 4a_1\phi_x}{2a_0}\right]_x + \left[\frac{b_{0x} + 4b_1\phi_x}{2b_0}\right]_x = 0. \tag{22a}$$

Without loss of generality, the nature of the equation (22*a*) permits us the freedom to choose

$$\left[\frac{a_{0x} + 4a_1\phi_x}{2a_0}\right] = f_1(t) = \mathrm{i}\lambda \qquad \left[\frac{b_{0x} + 4b_1\phi_x}{2b_0}\right] = f_2(t) = -\mathrm{i}\lambda \tag{22b}$$

where the parameter λ can now be a function of time. Introducing the squared eigenfunctions in the form $a_0 = \psi_1^2$, $b_0 = \psi_2^2$, $\phi_x = \psi_1\psi_2$, $a_1 = -u/2$ and $b_1 = -\epsilon u/2$, we get

$$\psi_x = U\psi \qquad \psi = (\psi_1\psi_2)^T \qquad U = \mathrm{i}\lambda\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \begin{bmatrix}0 & u\\ \pm u & 0\end{bmatrix}. \tag{23}$$

The time-dependent part of the Lax pair can be similarly explicitly worked out from equations (16*a*) and (17) using the transformation (21) in the form

$$\psi_t = V\psi \qquad V = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \tag{24}$$

where A, B and C have the following expressions:

$$\begin{aligned} A &= 4\mathrm{i}\lambda^3 x \pm 2\mathrm{i}\lambda\left(x\int u^2\,\mathrm{d}x\right)_x \\ B &= 4\lambda^2 xu - 2\mathrm{i}\lambda(xu)_x - (xu)_{xx} \pm 2u\left(x\int u^2\,\mathrm{d}x\right)_x \\ C &= \pm 4\lambda^2 xu \pm 2\mathrm{i}\lambda(xu)_x \mp (xu)_{xx} + 2u\left(x\int u^2\,\mathrm{d}x\right)_x. \end{aligned} \tag{25}$$

Compatibility of (23) and (24) requires that the spectral parameter should be time dependent and evolve as

$$\lambda_t = 4\lambda^3. \tag{26}$$

Solving (26) we have

$$\lambda(z,t) = \frac{1}{\sqrt{-8(t+z)}} \tag{27}$$

where z is a constant which can also serve as a spectral parameter that does not depend on the space-time coordinates. The spectral problem (23)–(27) is exactly the same as the one given by Burtsev *et al* [3], which is now obtained straightforwardly from the Painlevé analysis.

2.3. *Bäcklund transformation and soliton solutions*

Let us now generate (as an example) the soliton solutions of the deformed MKdV equation

$$u_t + (xu)_{xxx} + 2\left[\left(x\int u^2\,\mathrm{d}x\right)_x u\right]_x = 0. \tag{28}$$

For this purpose, we define the function

$$\Gamma = \frac{\psi_1}{\psi_2} \tag{29}$$

so that the AKNS system and its time evolution equations (23)–(25) are equivalent to the Riccati equations

$$\Gamma_x = 2\mathrm{i}\lambda\Gamma + u + u\Gamma^2 \tag{30}$$

$$\Gamma_t = 2A\Gamma + B - C\Gamma^2. \tag{31}$$

Now to construct the Bäcklund transformation [6], we define a new function $\Gamma' = 1/\Gamma$ satisfying equation (30) with a potential $u'(x)$ defined by

$$u' = u - 2\partial_x \tan^{-1}\Gamma. \tag{32}$$

Taking the trivial input solution $u = 0$, we can obtain the one-soliton solution as

$$u' = 2\eta_1\,\mathrm{sech}(2\eta_1 x + \delta_1) \tag{33a}$$

where

$$\eta_1 = \mathrm{i}\lambda \qquad \eta_{1t} = -4\eta_1^3 \tag{33b}$$

and δ_1 is a constant. With a suitable scale change of variables, the standard MKdV soliton can be recovered from equation (33). The procedure can be extended to obtain multisoliton solutions.

3. Deformations of the nonlinear Schrödinger (NLS) equation

The linearly x-dependent NLS equation

$$\mathrm{i}q_t + (\gamma_2 + \mu_2 x)(q_{xx} + 2|q|^2 q) + 2\mu_2\left(q_x + q\int_{-\infty}^{x} |q|^2\,\mathrm{d}x'\right) = 0 \tag{34}$$

and the radially symmetric NLS equation

$$\mathrm{i}q_t + q_{rr} + 2|q|^2 q + \frac{1}{r}q_r - \frac{1}{r^2}q + 4q\int_0^r \frac{1}{r'}|q|^2\,\mathrm{d}r' = 0 \tag{35}$$

as well as their geometrically/gauge equivalent spin systems have already been shown to be completely integrable and their corresponding Lax pairs have been derived through Painlevé analysis in [7, 8]. So we consider the cylindrical NLS equation alone in this section.

3.1. Cylindrical nonlinear Schrödinger (NLS) equation

We consider the evolution equation

$$\mathrm{i}\left(q_t + \frac{q}{2t}\right) + q_{xx} + 2|q|^2 q = 0. \tag{36}$$

Equation (36) describes cylindrically diverging quasiplane envelope waves in a nonlinear medium. In order to analyse this equation from the singularity structure point of view, we put $q = a$ and $q^* = b$ and rewrite equation (36) and its complex conjugate in the form

$$\mathrm{i}\left(a_t + \frac{a}{2t}\right) + a_{xx} + 2a^2 b = 0 \tag{37a}$$

$$-\mathrm{i}\left(b_t + \frac{b}{2t}\right) + b_{xx} + 2ab^2 = 0. \tag{37b}$$

Assuming the leading orders of the solutions of equation (37) to have the form

$$a = a_0\phi^{\alpha} \qquad b = b_0\phi^{\beta} \tag{38}$$

we obtain

$$\alpha = \beta = -1 \qquad a_0 b_0 = -\phi_x^2. \tag{39}$$

To find the resonances, we substitute the Laurent expansion

$$a = a_0\phi^{-1} + \cdots + a_j\phi^{j-1} + \cdots \qquad b = b_0\phi^{-1} + \cdots + b_j\phi^{j-1} + \cdots \tag{40}$$

into equation (37) and equate the coefficients of (ϕ^{j-3}, ϕ^{j-3}) to zero to give

$$\begin{bmatrix} (j-1)(j-2)\phi_x^2 & 2a_0^2 \\ 2b_0^2 & (j-1)(j-2)\phi_x^2 \end{bmatrix}\begin{bmatrix} a_j \\ b_j \end{bmatrix} = 0. \tag{41}$$

Solving (41), we get the resonance values

$$j = -1, 0, 3, 4. \tag{42}$$

Again $j = -1$ represents the arbitrariness of the singular manifold $\phi(x, t) = 0$ while equation (39) shows the arbitrariness of either a_0 or b_0. By collecting the coefficients of (ϕ^0, ϕ^0) and (ϕ^1, ϕ^1), one can show that either a_3 or b_3 and a_4 or b_4 is arbitrary along the same lines as in the case of deformed MKdV equation while the coefficients of (ϕ^{-2}, ϕ^{-2}) and (ϕ^{-1}, ϕ^{-1}) uniquely determine the pairs of coefficients (a_1, b_1) and (a_2, b_2). Thus equation (37) satisfies the Painlevé property.

3.2. Lax pair and soliton solutions

As before, we truncate the Laurent series at the constant level term to give

$$a = a_0\phi^{-1} + a_1 \qquad b = b_0\phi^{-1} + b_1 \tag{43}$$

where a_1 and b_1 satisfy equation (37). Substituting (43) into equation (37) and equating the coefficients of different powers of ϕ, we get the following set of differential equations:

(ϕ^{-3}, ϕ^{-3}):

$$a_0 b_0 = -\phi_x^2. \tag{44a}$$

(ϕ^{-2}, ϕ^{-2}):

$$\begin{aligned} &-\mathrm{i}a_0\phi_t - 2a_{0x}\phi_x - a_0\phi_{xx} + 2a_0^2 b_1 + 4a_0 b_0 a_1 = 0 \\ &+\mathrm{i}b_0\phi_t - 2b_{0x}\phi_x - b_0\phi_{xx} + 2a_1 b_0^2 + 4a_0 b_0 b_1 = 0. \end{aligned} \tag{44b}$$

(ϕ^{-1}, ϕ^{-1}):

$$\begin{aligned} &\mathrm{i}a_{0t} + \frac{\mathrm{i}a_0}{2t} + a_{0xx} + 2[a_1^2 b_0 + 2a_0 a_1 b_1] = 0 \\ &-\mathrm{i}b_{0t} - \frac{\mathrm{i}b_0}{2t} + b_{0xx} + 2[b_1^2 a_0 + 2b_0 a_1 b_1] = 0. \end{aligned} \tag{44c}$$

Differentiating (44a) with respect to time t, we get

$$a_{0t}b_0 + a_0 b_{0t} = -2\phi_x\phi_{xt}. \tag{45}$$

Making use of (44b) and (44c) in (45) and after simplification, we are now led to the following equation

$$\begin{aligned} \frac{\phi_x^2}{t} - 2\mathrm{i}[a_0^2 b_1^2 - b_0^2 a_1^2] = {}& \mathrm{i}[a_{0xx}b_0 - b_{0xx}a_0] - \frac{2\mathrm{i}\phi_{xx}}{\phi_x}[a_{0x}b_0 - b_{0x}a_0] \\ &+ 2\mathrm{i}\phi_x[a_{1x}b_0 + a_1 b_{0x} - b_{1x}a_0 - b_1 a_{0x}]. \end{aligned} \tag{46}$$

Rearranging equation (46) and after a little manipulation, we obtain

$$\mathrm{i}\left[\frac{a_{0x} + 2a_1\phi_x}{a_0}\right]_x = -\frac{1}{2t} \qquad \mathrm{i}\left[\frac{b_{0x} + 2b_1\phi_x}{b_0}\right]_x = \frac{1}{2t}. \tag{47}$$

Then defining $a_0 = \mathrm{i}\psi_1^2$, $b_0 = -\mathrm{i}\psi_2^2$, $\phi_x = \mathrm{i}\psi_1\psi_2$, $a_1 = -\mathrm{i}q$ and $b_1 = \mathrm{i}q^*$, we get the space part of the eigenvalue problem

$$\psi_x = U\psi \qquad U = \mathrm{i}\lambda\sigma_3 + \mathrm{i}\begin{bmatrix} 0 & q \\ q^* & 0 \end{bmatrix} \tag{48}$$

and the spectral parameter defined above is a function of both space and time variables and is of the form

$$\lambda(\mu, x, t) = \frac{(\mu + x/4)}{t}. \tag{49}$$

As before, the space-time dependence in the spectral parameter can be avoided provided it is replaced by equation (49) with μ serving as a spectral parameter. Similarly, the time part of the Lax pair can be worked out as in the previous case and is of the form

$$\psi_t = V\psi \tag{50a}$$

where V has the same form as (24) with its components now having the following form

$$A = -2\mathrm{i}\lambda^2 + \mathrm{i}|q|^2 \qquad B = -2\mathrm{i}\lambda q - q_x \qquad C = -2\mathrm{i}\lambda q^* + q_x^*. \tag{50b}$$

To generate soliton solutions, one can proceed as in section 2.3 and obtain the Bäcklund transformation (BT)

$$q'(x) = q(x) + \frac{2(\Gamma^2\Gamma_x^* - \Gamma_x)}{1 - |\Gamma|^4} \tag{51}$$

where again the function Γ is of the same form as equation (29). The one-soliton solution turns out to be

$$q(1) = -\frac{2\beta}{t}\operatorname{sech}\left[\frac{8\alpha\beta + 2\beta x}{t} + \delta_1\right]\exp\left[\frac{4\mathrm{i}}{t}\left(\alpha^2 - \beta^2 + \frac{x^2}{16} + \frac{\alpha x}{2}\right)\right] \tag{52}$$

where α and β are constants.

4. Deformed KdV equation

The deformation of the KdV equation has the form [3]

$$u_t = \tfrac{1}{2}Q_{xxx} + 2Q_x u + Q u_x \tag{53a}$$

$$Q = -\frac{1}{\sqrt{x}}\int\sqrt{x}\left(2u_x + \frac{4u}{x}\right)\mathrm{d}x. \tag{53b}$$

Under the transformation

$$u(x,t) = -\frac{5}{16x^2} + \hat{u}(\hat{x},\hat{t}) \qquad \hat{x} = x^{3/2} \qquad \hat{t} = \frac{27}{8}t \tag{54}$$

the system (53) is reduced to the form

$$u_t + (xu)_{xxx} + u_{xx} = \left[-3xu^2 - 2u\int u\,\mathrm{d}x\right]_x - 3u^2 \tag{55}$$

where the hats have been dropped for convenience.

4.1. Painlevé analysis

To analyse the singularity structure aspects of equation (55), we put $u = v_x$ so that (55) becomes

$$v_{xt} + 4v_{xxx} + xv_{xxxx} + 8v_x^2 + 2v_{xx}v + 6xv_xv_{xx} = 0. \tag{56}$$

To bring out the leading order behaviour of the above equation, we put

$$v = v_0\phi^\alpha \tag{57}$$

in equation (56) and obtain

$$\alpha = -1 \qquad v_0 = 2\phi_x. \tag{58}$$

To find the resonances, we substitute the Laurent series

$$v = v_0\phi^{-1} + \cdots + v_j\phi^{j-1} + \cdots \tag{59}$$

in (56) and compare the coefficients of ϕ^{j-5} to give

$$(j-1)[(j-2)(j-3)(j-4) - 12(j-2) + 24] = 0. \tag{60}$$

Solving (60), we get

$$j = -1, 1, 4, 6. \tag{61}$$

As before, in the above Laurent series v_1, v_4 and v_6 can be proved to be arbitrary. Hence the system (53) through (56) is expected to be integrable.

4.2. Lax pair

To derive the Lax pair, we substitute the truncated Laurent series

$$v = v_0\phi^{-1} + v_1 \tag{62}$$

in (56) and compare the coefficients of various powers of ϕ to give the following set of equations:

ϕ^{-5}:

$$v_0 = 2\phi_x \tag{63a}$$

ϕ^{-4}:

$$\begin{aligned}&-24v_0\phi_x^3 - 24xv_{0x}\phi_x^3 - 36xv_0\phi_x^2\phi_{xx} + 12v_0^2\phi_x^2 + 24xv_0v_{0x}\phi_x^2\\ &+6xv_0^2\phi_x\phi_{xx} = 0\end{aligned} \tag{63b}$$

ϕ^{-3}:

$$\begin{aligned}&4\phi_x^2\phi_t + 48\phi_x^2\phi_{xxx} - 10\phi_x^2\phi_{xx} - 3x\phi_x\phi_{xx}^2 + 2\phi_x^3v_1 + 4x\phi_x^2\phi_{xxx}\\ &+6xv_{1x}\phi_x^3 = 0\end{aligned} \tag{63c}$$

ϕ^{-2}:

$$\begin{aligned}&-2\phi_{xx}\phi_t - 4\phi_x\phi_{xt} - 24\phi_{xxx}\phi_x - 24\phi_{xx}^2 - 10x\phi_x\phi_{xxxx} + 4x\phi_{xx}\phi_{xxx}\\ &+32\phi_x^2 - 32\phi_x^2v_{1x} - 12v_1\phi_x\phi_{xx} - 12x\phi_x^2v_{1xx} - 36x\phi_x\phi_{xx}v_{1x} = 0\end{aligned} \tag{63d}$$

ϕ^{-1}:

$$\begin{aligned}&\phi_{xxt} + 4\phi_{xxxx} + x\phi_{xxxxx} + 16\phi_{xx}v_{1x} + 2\phi_xv_{1xx} + 2v_1\phi_{xxx} + 6x\phi_{xx}v_{1xx}\\ &+6xv_{1x}\phi_{xxx} = 0.\end{aligned} \tag{63e}$$

Evaluating ϕ_{xt} from (63*c*) and (63*d*) (equation (63*b*) is just an identity), one arrives at the following equation after simplification

$$\frac{8\phi_{xx}^2}{\phi_x} - 8\phi_x + \frac{3x\phi_{xx}\phi_{xxx}}{\phi_x} - \frac{3}{2}x\frac{\phi_{xx}^3}{\phi_x^2} - \frac{3}{2}x\phi_{xxxx} - 3xv_{1xx}\phi_x = 0. \tag{64}$$

Letting

$$\phi_x = \psi^2 \tag{65}$$

and substituting (65) in (64), we obtain

$$32\psi_x^2 - 8\psi^2 + 3x\psi_x\psi_{xx} - 3x\psi\psi_{xxx} - 3xv_{1xx}\psi^2 = 0. \tag{66}$$

Rearranging equation (66) and expressing the spectral parameter λ in terms of the eigenfunction of the required form, we obtain the spatial part of the eigenvalue problem of equation (53) through equation (56) as

$$\psi_{xx} + (u + \lambda)\psi = 0 \tag{67}$$

where now the spectral parameter is a function of space and time. The time part of the Lax pair of equation (53) can be worked out in a similar fashion from equations (63*c*, *d*) and (65) and it has the form

$$\psi_t = A\psi_x + B\psi \tag{68}$$

where

$$A = 4\lambda + Q \qquad B = -\tfrac{1}{2}A_x. \tag{69}$$

The compatibility condition of equations (67) and (68) requires that the spectral parameter should evolve as

$$\lambda_x = \frac{\lambda}{x} \qquad \lambda_t = \frac{12}{x}\lambda^2. \tag{70}$$

The soliton solutions can be generated using Bäcklund transformations as before.

Finally, the inhomogeneous Kaup system [3, 9] given by

$$u_t + \tfrac{1}{2}(xu^2)_x + (x\eta)_x = -\eta \tag{71a}$$

$$\eta_t + (x\eta u)_x + (xu)_{xxx} = -\eta u \tag{71b}$$

can again be investigated through Painlevé analysis and the Lax pair can also be derived on the same lines as before [10]. The results are in conformity with [3].

5. Conclusion

In this paper, we have investigated the singularity structure aspects of the deformations of various well known (1 + 1)-dimensional integrable equations discussed by Burtsev *et al* [3] in the appendix of their paper and showed that all these specific equations satisfy the Painlevé property. We have also generated the Lax pair straightforwardly from the Painlevé analysis and used them to generate soliton solutions. Further, Burtsev *et al* [3] and Burtsev and Gavitov [11] have pointed out that in general the variable spectral parameters themselves satisfy nonlinear partial differential equations, which may be non-Painlevé in nature. As an example, the Maxwell–Bloch equation with constant coefficients is cited in [3], whose variable spectral parameter satisfies a PDE which on similarity reduction is stated to admit movable critical points [11]. On the other hand, considering the same Maxwell–Bloch equation with constant coefficients,

$$E_\eta = \rho \qquad N_\xi + \tfrac{1}{2}(\rho^* E + \rho E^*) = c \qquad \rho_\xi = NE \tag{72}$$

an explicit analysis of the Laurent series of the solution as was done in the previous sections shows that the resonances occur at

$$j = -1, 0, 2, 3, 4 \tag{73}$$

and further analysis shows that the Painlevé property is indeed satisfied. So it appears that the Painlevé property of the equations defining the spectral parameter need not affect the Painlevé property of the integrable deformations of the original nonlinear evolution equations. At least, we have not come across any example contrary to this, and we believe that in integrable deformations of the evolution equations, the Painlevé property will hold.

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